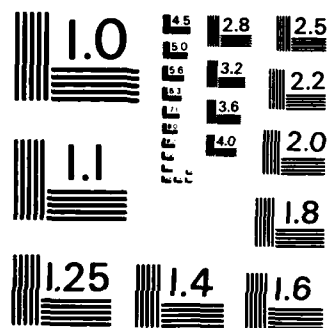END

FILMED

DTIC

MICROCOPY RESOLUTION TEST CHART

NATIONAL BUREAU OF STANDARDS-1963-A

AD-A161 293

ARO /9367.30-MA

REPORT DOCUMENTATION PAGE

REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
M693 D-76		

TITLE (and subtitle)	5. TYPE OF REPORT & PERIOD COVERED
On The Limit Behavior Of A Multi-Compartment Storage Model With An Underlying Markov Chain I: Without Normalization	Technical
	6. PERFORMING ORG. REPORT NUMBER

AUTHOR(s)	8. CONTRACT OR GRANT NUMBER(s)
Eric S. Tollar	USARO No. DAAG 29-82-K-0168

PERFORMING ORGANIZATION NAME AND ADDRESS	10. PROGRAM ELEMENT, PROJECT, TASK AREA, & WORK UNIT NUMBERS
The Florida State University Department of Statistics Tallahassee, FL 32306-3033	

CONTROLLING OFFICE NAME AND ADDRESS	12. REPORT DATE
U.S. Army Research Office - Durham P.O. Box 12211 Research Triangle Park, NC 27709	February 1985
	13. NUMBER OF PAGES
	17

MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report)
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

DISTRIBUTION STATEMENT (of this report)

approved for public release; distribution unlimited

DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from report)

SUPPLEMENTARY NOTES

KEY WORDS

Storage model; Markov chain; Dual Markov chain; Auxillary Markov chain,

ABSTRACT (Continue on reverse side if necessary and identify by block number)

The present paper considers a multi-compartment storage model with one way flow. The inputs and outputs for each compartment are controlled by a denumerable state Markov chain. Assuming finite first moments, it is shown that certain compartments converge in distribution while others diverge, based on appropriate first moment conditions on the inputs and outputs.

DTIC FILE COPY

NOV 21 1985

85 11 19 054

ON THE LIMIT BEHAVIOR OF A MULTI-COMPARTMENT STORAGE MODEL
WITH AN UNDERLYING MARKOV CHAIN I: WITHOUT NORMALIZATION

by

Eric S. Tollar

February, 1985

The Florida State University
Department of Statistics
Tallahassee, Florida 32306

On the Limit Behavior of a Multi-Compartment Storage Model

with an Underlying Markov Chain I: Without Normalization

by

Eric S. Tollar

ABSTRACT

The present paper considers a multi-compartment storage model with one way flow. The inputs and outputs for each compartment are controlled by a denumerable state Markov chain. Assuming finite first moments, it is shown that certain compartments converge in distribution while others diverge, based on appropriate first moment conditions on the inputs and outputs.

1. INTRODUCTION

Using a stochastic model to approximate the behavior of various physical storage systems has become wide spread (for example, see Moran [6], Prabhu [7]). Initially, the random mechanism underlying the models was assumed to be independent, but in 1965 Lloyd and Odoom [5] proposed a model in which a dependent structure was feasible by assuming an underlying Markov chain as part of the random mechanism. The stochastic model was later expanded to having an underlying semi-Markov process, and a specific one compartment storage model with underlying semi-Markov process was considered by Puri [8], Puri and Senturia [9], [10], Balagopal [1], Puri and Woolford [11], and others.

In this paper, we will consider a multi-compartment storage system with one-way flow, similar to that of Puri and Senturia. However, we will only consider the model with an underlying Markov chain; the more general case for semi-Markov processes to be considered in a later paper. In this model, material will flow into the system via compartment 1. Each of the subsequent compartments will get material from its immediate predecessor by "demanding" a certain amount of material. The previous compartment will then transfer the material demanded, or all the material in the compartment, depending on which is smaller. Finally, the system will lose material by "demands" placed on the last compartment, which are dealt with as above.

In section 2 we will give the mathematical formulation of the model, and the closed form for the amount in each compartment. In section 3 we will establish the limit behavior of the compartments without normalization.

2. THE MODEL

Let $\{X_n, n = 0, 1, \ldots\}$ be an aperiodic, irreducible, positive recurrent Markov chain with state space J, where we take J to be denumerable. We denote the elements of J by $\{1, 2, \ldots\}$. Define the transition matrix for the Markov chain $\underset{\sim}{P} = (p_{ij})$ by

$$P(X_n = j | X_{n-1} = i) \equiv p_{ij}, \qquad (2.1)$$

and let $\underset{\sim}{\pi} = (\pi_1, \pi_2, \ldots)$ be the stationary probability measure satisfying $\underset{\sim}{\pi}\underset{\sim}{P} = \underset{\sim}{\pi}$. Further, we define the number of visits to stage $j \in J$ in n steps by

$$M_j(n) = \sum_{i=0}^{n} I(X_i = j) \qquad (2.2)$$

where $I(A)$ is the indicator function of set A.

For the k-compartment model to be considered, the transmission of material in the system will be controlled by the underlying Markov chain $\{X_n\}$ and a collection of infinite $k + 1$ - tuples governing the transfer of material. These $k + 1$ - tuples are defined as follows. For each $i \in J$, let $\{\underset{\sim}{V}_n(i) = (V_{0,n}(i), V_{1,n}(i), \ldots, V_{k,n}(i)): n = 1, 2, \ldots\}$ be a sequence of i.i.d. $k + 1$ - tuples, independent of $\{X_n\}$ and $\{\underset{\sim}{V}_n(j)\}$ for $j \neq i$. We will insist that the following conditions hold.

i) $P(\underset{\substack{\ell \neq j \\ 0 \leq \ell, j \leq k}}{\cup} \{V_{\ell,n}(i) \neq 0, V_{j,n}(i) \neq 0\}) = 0, \; \forall i, \; \forall n > 0.$

$$(2.3)$$

ii) $P(V_{j,n}(i) < 0) = 0 \, \forall i, \; \forall j, \; \forall n > 0.$

As will be seen later, the assumptions (2.3) are not mathematically necessary. However, it is not clear that the model itself would be reasonable without them. Without (2.3i), we would have the problem of simultaneous transfers in

more than one compartment, and it is not clear what should be done in this case. Without (2.3ii) we would not have a one-way flow model, and it would be there-fore possible to have compartments with negative amounts of material in them.

To define the equations for the total amount of material in each of the k compartments, we first define $\underset{\sim}{C}(n) = (C_0(n), C_1, (n), \ldots, C_k(n))$, where $C_i(n)$ represents the total amount of material that has left compartment i by step n. Then it is easy to see that the amount of material in the various compartments, denoted by $\underset{\sim}{Z}(n) = (Z_1(n), \ldots, Z_k(n))$, is governed by the relation

$$Z_i(n) = C_{i-1}(n) - C_i(n).$$
(2.4)

We define $\underset{\sim}{C}(n)$ recursively by

$$C_0(n) = \sum_{i=1}^{n} V_{0,i}(X_i) + C_0^*,$$
(2.5)

$$C_i(n) = \begin{cases} C_i^*, & \text{for } n = 0 \\ \min[C_{i-1}(n), V_{i,n}(X_i) + C_i(n-1)], & \text{for } n > 0, \end{cases}$$
(2.6)

where $C_0^*, C_1^*, \ldots, C_k^*$ are the initial values of $\underset{\sim}{C}(0)$ (possibly random), satisfying

$$C_0^* \geq C_1^* \geq \ldots \geq C_k^* \quad \text{a.s.}$$
(2.7)

By examination of equations (2.4) through (2.7), it is easy to see the system will function as explained in the introduction.

To shorten the expressions for subsequent theorems, the following notation is introduced.

$$S_\ell(n) \begin{cases} 0, & \text{for } n = 0 \\ \sum_{i=1}^{n} V_{\ell,i}(X_i) + C_i^*, & \text{for } n > 0. \end{cases}$$

THEOREM 2.1. The following relationship holds for $1 \leq i \leq k$, $n \geq 1$.

$$C_i(n) = \min_{0 \leq j_1 \leq \ldots \leq j_i \leq n} (S_0(j_1) + [S_1(j_2) - S_1(j_1)] + \ldots + [S_i(n) - S_i(j_i)])$$

The proof will be omitted, as it is straightforward by induction, establishing the validity for each cell i by assuming it is true for the previous cells.

COROLLARY: The following relation holds for $1 \leq i \leq k$, $n > 0$:

$$Z_i(n) = \min_{0 \leq j_1 \leq \ldots \leq j_{i-1} \leq n} [S_0(j_1) + \ldots + (S_{i-1}(n) - S_{i-1}(j_{i-1}))]$$

$$- \min_{0 \leq j_1 \leq \ldots \leq j_i \leq n} [S_0(j_1) + \ldots + (S_i(n) - S_i(j_i))].$$

The proof is omitted, as it follows immediately from the theorem.

In the next section, we establish the limit behavior of $\underset{\sim}{Z}(n)$ without normalization.

3. LIMIT BEHAVIOR OF THE PROCESS

In this section we establish the limit behavior of the process $\underset{\sim}{Z}(n)$ without any normalization, when a first moment condition is assumed. The necessary moments are the standard ones for processes defined on a Markov chain. That is, if $\underset{\sim}{\pi}$ is the stationary measure for the Markov chain and if for every $j \in J$, $\{Y_n(j)\}$ is an i.i.d. sequence for each j, independent of the Markov chain and $\{Y_n(i)\}$ if $i \neq j$, then we define

$$E_\pi Y = \sum_{j \in J} \pi_j EY_1(j). \tag{3.1}$$

For $1 \leq \ell \leq k$ we will refer to compartment ℓ as subcritical, critical or

supercritical when $E_\pi V_\ell - \min\limits_{0 \le i < \ell} (E_\pi V_i)$ is greater than, equal to, or less than

zero, respectively. It will be established that as n tends to infinity, the

subcritical compartments converge, while the critical and supercritical compart-

ments diverge.

The following definitions will also be used in this section.

Definition: $\{\hat{X}_n\}$ is called the <u>dual Markov chain</u> of $\{X_n\}$ if

1) $P(\hat{X}_0 = j) = \pi_j$ for all j, and

2) $P(\hat{X}_{n+1} = i | \hat{X}_n = j) = \pi_j^{-1} \pi_i p_{ij}$.

As has been shown (see Çinlar [3]), we have that $\{\hat{X}_n\}$ is essentially the "reverse"

of $\{X_n\}$, that is, if X_0 has distribution $\underset{\sim}{\pi}$,

$$P(X_m = i_0, \ldots, X_{m+n} = i_n) = P(\hat{X}_m = i_n, \ldots, \hat{X}_{m+n} = i_0). \tag{3.2}$$

We will define

$$\hat{Z}_i(n) = \min_{0 \le j_1 \le \ldots \le j_{i-1} \le n} (\hat{S}_{i-1}(j_1) + [\hat{S}_{i-2}(j_2) - \hat{S}_{i-2}(j_1)] + \ldots + [\hat{S}_0(n) - \hat{S}_0(j_{i-1})])$$

$$\tag{3.3}$$

$$- \min_{0 \le j_1 \le \ldots \le j_i \le n} (\hat{S}_i(j_1) + [\hat{S}_{i-1}(j_2) - \hat{S}_{i-1}(j_1)] + \ldots + [\hat{S}_0(n) - \hat{S}_0(j_i)]),$$

where for given n,

$$\hat{S}_\ell(m) = \begin{cases} \sum\limits_{i=1}^{m} V_{\ell,i}(\hat{X}_i) & \text{for } m < n \\ \\ \sum\limits_{i=1}^{m} V_{\ell,i}(\hat{X}_i) + C_\ell^* & \text{for } m = n. \end{cases} \tag{3.4}$$

It can then be easily shown if $X_0 \sim \underset{\sim}{\pi}$ that

$$P(Z_1(n) \le x_1, \ldots, Z_k(n) \le x_k) = P(\hat{Z}_1(n) \le x_1, \ldots, \hat{Z}_k(n) \le x_k). \tag{3.5}$$

Definition: $\{\bar{X}_n\}$ is called the <u>auxiliary Markov chain</u> of $\{X_n\}$ if $\bar{X}_0$ has initial distribution $\underset{\sim}{\pi}$, and transition probabilities as in (2.1). As shown in Hoel, *et al.* [4], if we define

$$T = \min_{i > 0} \{X_i = \bar{X}_i\}, \tag{3.6}$$

then $T < \infty$ a.s. and

$$P(X_{M+1} = i_1, \ldots, X_{M+n} = i_n, T \le M) = P(\bar{X}_{M+1} = i_1, \ldots, \bar{X}_{M+n} = i_n, T \le M). \tag{3.7}$$

We will also define

$$\bar{S}_\ell(n) = \begin{cases} 0, & \text{for } n = 0 \\ \sum_{i=1}^{n} V_{\ell,i}(\bar{X}_i) + C_i^*, & \text{for } n > 0, \end{cases} \tag{3.8}$$

and define

$$\bar{Z}_i(n) = \min_{0 \le j_1 \le \ldots \le j_{i-1} \le n} (\bar{S}_0(j_1) + [\bar{S}_1(j_2) - \bar{S}_1(j_1)] + \ldots + [\bar{S}_{i-1}(n) - \bar{S}_{i-1}(j_{i-1})])$$

$$\tag{3.9}$$

$$- \min_{0 \le j_1 \le \ldots \le j_i \le n} (\bar{S}_0(j_1) + [\bar{S}_1(j_2) - \bar{S}_1(j_1)] + \ldots + [\bar{S}_i(n) - \bar{S}_i(j_i)]).$$

The main theorem to be established in this section is the following:

THEOREM 3.1. <u>For any arbitrary distribution of</u> X_0 <u>and</u> $\underset{\sim}{Z}(0)$, <u>if</u> $E_\pi |V_j| < \infty$ <u>for</u> $0 \le j \le k$, <u>then as</u> $n \to \infty$, $P(Z_1(n) \le x_1, \ldots, Z_k(n) \le x_k) \to P(Z_1 \le x_1, \ldots, Z_k \le x_k)$ <u>for all continuity points</u> $(x_1, \ldots, x_k)$ <u>of the distribution of some random variables</u> $Z_1, \ldots, Z_k$ <u>where</u>

$$P(Z_j < \infty) = \begin{cases} 0 \text{ if } E_\pi V_j \le \min_{0 \le i < j} (E_\pi V_i) \\ 1 \text{ if } E_\pi V_j > \min_{0 \le i < j} (E_\pi V_i). \end{cases}$$

The proof of this theorem requires several steps, which are broken down into the lemmas and theorems that follow. The first such lemma, which is stated without proof, is a straightforward extension of the well known result that

$$\lim_{n \to \infty} \max_{0 \le j \le n} (\sum_{i=0}^{j} Y_i(X_i)) = Z^* \text{ a.s., where } P(Z^* < \infty) = 0 \cdot I(E_\pi Y \ge 0) + 1 \cdot I(E_\pi Y < 0),$$

for $\{Y_n(j)\}$ as in (3.1).

LEMMA 3.2. _If for all_ $j \in J$, $\{\underset{\sim}{Y}_n(j) = (Y_{1,n}(j), \ldots, Y_{\ell,n}(j)): n \ge 1\}$ _is an i.i.d. sequence, independent of_ $\{\underset{\sim}{Y}_n(i)\}$ _for_ $i \ne j$, _and if_ $E_\pi |Y_i| < \infty$ _for_

$1 \le i \le \ell$, _then_ $\lim_{n \to \infty} \quad \max_{0 \le j_1 \le \ldots \le j_\ell \le n} (\sum_{k=1}^{\ell} \sum_{i=1}^{j_k} Y_{k,i}(X_i)) \equiv Z^*$ _exists almost surely,_

where

$$P(Z^* < \infty) = \begin{cases} 1 \text{ \underline{if} } \sum_{i=j}^{\ell} E_\pi Y_i < 0 \text{ \underline{for all} } i, \ 1 \le i \le \ell \\ 0 \text{ \underline{otherwise}.} \end{cases}$$

The next theorem to be established is the following.

THEOREM 3.3. _Let_ X_0 _have initial distribution_ $\underset{\sim}{\pi}$, $E_\pi |V_j| < \infty$ _for_ $0 \le j \le k$. _Then for all initial distributions_ $\underset{\sim}{Z}(0)$, _as_ $n \to \infty$

$$P(Z_1(n) \le x_1, \ldots, Z_k(n) \le x_k) \to P(Z_1 \le x_1, \ldots, Z_k \le x_k)$$

for all continuity points $(x_1, \ldots, x_k)$ _of the distribution of_ $(Z_1, \ldots, Z_k)$, _where for_ $1 \le \ell \le k$,

$$P(Z_\ell < \infty) = 0 \cdot I(E_\pi V_\ell \leq \min_{0 \leq i < \ell} E_\pi V_i) + 1 \cdot I(E_\pi V_\ell > \min_{0 \leq i < \ell} E_\pi V_i).$$

PROOF. From (3.5), we have that we need only consider $(\hat{Z}_1(n), \ldots, \hat{Z}_k(n))$ to complete the proof. The almost sure behavior of each $\hat{Z}_i(n)$ is established as $n \to \infty$; the joint behavior then follows automatically. From (3.3) we can see that for any i,

$$
\begin{aligned}
\hat{Z}_i(n) &= \max_{0 \leq j_1 \leq \ldots \leq j_i \leq n} (-\hat{S}_i(j_1) - \ldots - [\hat{S}_0(n) - \hat{S}_0(j_i)]) \\
&\quad - \max_{0 \leq j_1 \leq \ldots \leq j_{i-1} \leq n} (-\hat{S}_{i-1}(j_1) - \ldots - [\hat{S}_0(n) - \hat{S}_0(j_{i-1})]) \\
&= \max_{0 \leq j_1 \leq \ldots \leq j_i \leq n} (\sum_{k=1}^{i} [\hat{S}_{i-k}(j_k) - \hat{S}_{i-k+1}(j_k)] - \hat{S}_0(n)) \\
&\quad - \max_{0 \leq j_1 \leq \ldots \leq j_{i-1} \leq n} (\sum_{k=1}^{i-1} [\hat{S}_{i-k-1}(j_k) - \hat{S}_{i-k}(j_k)] - \hat{S}_0(n)) \\
&= \max_{0 \leq j_1 \leq \ldots \leq j_i \leq n} (\sum_{k=1}^{i} [\hat{S}_{i-k}(j_k) - \hat{S}_{i-k+1}(j_k)]) \\
&\quad - \max_{0 \leq j_1 \leq \ldots \leq j_{i-1} \leq n} (\sum_{k=1}^{i-1} [\hat{S}_{i-k-1}(j_k) - \hat{S}_{i-k}(j_k)]).
\end{aligned}
$$

(3.10)

By defining

$$R_m(i) = \hat{S}_{m-1}(i) - \hat{S}_m(i) \tag{3.11}$$

(3.10) reduces to

$$
\begin{aligned}
\hat{Z}_i(n) &= \max_{0 \leq j_1 \leq \ldots \leq j_i \leq n} [\sum_{k=1}^{i} R_{i-k+1}(j_k)] \\
&\quad - \max_{0 \leq j_1 \leq \ldots \leq j_{i-1} \leq n} [\sum_{k=1}^{i-1} R_{i-k}(j_k)].
\end{aligned}
\tag{3.12}
$$

For $i = 1$, we have for all initial distributions of $Z_1(0)$, as $n \to \infty$,

$$\hat{Z}_1(n) = \max_{0 \le j \le n} (R_1(j)) \to Z_1, \text{ a.s.,}$$

where

$$P(Z_1 < \infty) = 0 \cdot I(E_\pi V_1 \le E_\pi V_0) + 1 \cdot I(E_\pi V_1 > E_\pi V_0).$$

(For details, see Puri and Woolford [11].) Since this is the condition desired for $i = 1$, we proceed by induction, assuming the theorem holds for $\ell \le i$ and showing it is true for $i + 1$.

Let

$$w = \max_{0 \le \ell \le i} \{\ell : E_\pi V_\ell = \min_{0 \le j \le i} (E_\pi V_j)\} \tag{3.13}$$

Then from (3.12) we have

$$\hat{Z}_{i+1}(n) = \max_{0 \le j_1 \le \ldots \le j_{i+1} \le n} [\sum_{k=1}^{i+1} R_{i-k+2}(j_k)] - \sum_{\ell=w+1}^{i} \hat{Z}_\ell(n)$$

$$- \max_{0 \le j_1 \le \ldots \le j_w \le n} [\sum_{k=1}^{w} R_{w-k+1}(j_k)].$$

Since for $w + 1 \le \ell \le i$, $E_\pi V_\ell > \min_{0 \le k < \ell} (E_\pi V_k) = E_\pi V_w$, by the induction hypothesis $\hat{Z}_\ell(n) \to Z_\ell$ a.s., where $P(Z_\ell < \infty) = 1$. Consequently, the behavior of $\hat{Z}_{i+1}(n)$ depends upon the term

$$\hat{Y}_{i+1}(n) = \max_{0 \le j_1 \le \ldots \le j_{i+1} \le n} [\sum_{k=1}^{i+1} R_{i-k+2}(j_k)]$$

$$\tag{3.14}$$

$$- \max_{0 \le j_1 \le \ldots \le j_w \le n} [\sum_{k=1}^{w} R_{w-k+1}(j_k)].$$

Note that if $w = 0$ (3.15) reduces to

$$\hat{Y}_{i+1}(n) = \max_{0 \le j_1 \le \ldots \le j_{i+1} \le n} [\sum_{k=1}^{i+1} R_{i-k+2}(j_k)], \qquad (3.15)$$

which has the desired properties, as can be easily established by appealing to Lemma 3.2 (after rewriting (3.15) as in the lemma). Thus, assume $w \ge 1$.

Clearly for all $M < n$

$$\max_{0 \le j_1 \le \ldots \le j_{i-w+1} \le M} (\sum_{k=1}^{i-w+1} R_{i-k+2}(j_k))$$

$$+ \max_{M \le j_1 \le \ldots \le j_w \le n} (\sum_{k=1}^{w} R_{w-k+1}(j_k))$$

$$- \max_{0 \le j_1 \le \ldots \le j_w \le n} (\sum_{k=1}^{w} R_{w-k+1}(j_k)) \qquad (3.16)$$

$$\le \hat{Y}_{i+1}(n)$$

$$\le \max_{0 \le j_1 \le \ldots \le j_{i-w+1} \le n} (\sum_{k=1}^{i-w+1} R_{i-k+2}(j_k)).$$

By Lemma 3.4, to be established later, we have for all $\omega \in \Omega$ and $M > 0$, there is an $N_M > M$ where for all $n > N_M$,

$$\max_{0 \le j_1 \le \ldots \le j_w \le n} (\sum_{k=1}^{w} R_{w-k+1}(j_k)) = \max_{M \le j_1 \le \ldots \le j_w \le n} (\sum_{k=1}^{w} R_{w-k+1}(j_k)). \qquad (3.17)$$

Thus, from (3.16) and (3.17), we get

$$\max_{0 \le j_1 \le \ldots \le j_{i-w+1} \le M} (\sum_{k=1}^{i-w+1} R_{i-k+2}(j_k))$$

$$\le \hat{Y}_{i+1}(n) \qquad (3.18)$$

$$\le \max_{0 \le j_1 \le \ldots \le j_{i-w+1} \le n} (\sum_{k=1}^{i-w+1} R_{i-k+2}(j_k)).$$

From Lemma 3.2, it is clear that the behavior of

$$\max_{0 \le j_1 \le \ldots \le j_{i-w+1} \le n} \left(\sum_{k=1}^{i-w+1} R_{i-k+2}(j_k) \right)$$

depends on $E_\pi V_w - E_\pi V_{i+1}$.

CASE I. $E_\pi V_w - E_\pi V_{i+1} < 0$: In this case, we have from Lemma 3.2 that

$$\max_{0 \le j_1 \le \ldots \le j_{i-w+1} \le n} \left(\sum_{k=1}^{i-w+1} R_{i-k+2}(j_k) \right) \to Y \text{ a.s.},$$

where $P(Y < \infty) = 1$, independent of the initial distribution. Thus, for almost all $\omega \in \Omega$, and $\varepsilon > 0$, we can choose an M where

$$\max_{0 \le j_1 \le \ldots \le j_{i-w+1} \le n} \left(\sum_{k=1}^{i-w+1} R_{i-k+2}(j_k) \right)$$

$$< \max_{0 \le j_1 \le \ldots \le j_{i-w+1} \le M} \left(\sum_{k=1}^{i-w+1} R_{i-k+2}(j_k) \right) + \varepsilon. \tag{3.19}$$

Consequently, it follows from (3.19) that $\hat{Y}_{i+1}(n) \to Y$ a.s.

CASE II. $E_\pi V_w - E_\pi V_{i+1} \ge 0$: In this case, for almost every $\omega \in \Omega$ and any $R > 0$, there is an M where

$$\max_{0 \le j_1 \le \ldots \le j_{i-w+1} \le M} \left(\sum_{k=1}^{i-w+1} R_{i-k+2}(j_k) \right) > R.$$

Thus, from (3.18) we get that $\hat{Y}_{i+1}(n) \to \infty$ a.s.. □

Now the Lemma cited in Theorem 3.3 is established.

LEMMA 3.4. *For ν as defined in (3.13), for almost all $\omega \in \Omega$, and for every $M > 0$,*

there exists an $N_M > M$ <u>such that for all</u> $n > N_M$,

$$\max_{0 \le j_1 \le \ldots \le j_w \le n} (\sum_{k=1}^{w} R_{w-k+1}(j_k)) = \max_{M \le j_1 \le \ldots \le j_w \le n} (\sum_{k=1}^{w} R_{w-k+1}(j_k)).$$

<u>PROOF</u>: First, we show that for every $M > 0$ there exists an $N_M > M$ where

$$\max_{0 \le j_1 \le \ldots \le j_w \le N,} (\sum_{k=1}^{w} R_{w-k+1}(j_k)) = \max_{M \le j_1 \le \ldots \le j_w \le N,} (\sum_{k=1}^{w} R_{w-k+1}(j_k)). \quad (3.20)$$

Since $E_\pi V_w \le \min_{0 \le j < w} E_\pi V_j$ (the inductive hypothesis in Theorem 3.3 is still considered to hold), we have that $\hat{Z}_w(n) \to \infty$ a.s.. However, if (3.20) is not true, then there exists an M where

$$\hat{Z}_{w,n} = \max_{\substack{0 \le j_1 \le \ldots \le j_w \le n \\ 0 \le j_1 < M}} (\sum_{k=1}^{w} R_{w-k+1}(j_k))$$

$$- \max_{0 \le j_1 \le \ldots \le j_{w-1} \le n} (\sum_{k=1}^{w-1} R_{w-k}(j_k)) \quad (3.21)$$

$$\le \max_{0 \le j_1 < M} (R_w(j_1)) < \infty.$$

Thus, we get that (3.20) must hold. We now show that if $M > 0$, then there is an N_M where for all $n > N_M$,

$$\max_{0 \le j_1 \le \ldots \le j_w \le n} (\sum_{k=1}^{w} R_{w-k+1}(j_k)) = \max_{M \le j_1 \le \ldots \le j_w \le n} (\sum_{k=1}^{w} R_{w-k+1}(j_k)). \quad (3.22)$$

First, for C_i^* as defined in (3.8), let $C_i^* = 0$, $0 \le i \le w$. Then by induction, it is enough to show that if (3.22) is true for n, then it must be true for $n + 1$. Assume (3.22) holds for n, and that for $M \le a_1 \le \ldots \le a_w \le n$,

$$\sum_{k=1}^{W} R_{w-k+1}(a_k) = \max_{0 \le j_1 \le \ldots \le j_w \le n} (\sum_{k=1}^{W} R_{w-k+1}(j_k)). \tag{3.23}$$

Show for indices where $0 \le b_1 \le \ldots \le b_w \le n+1$ and $b_1 < M$, there exist indices $\underset{\sim}{c}$

where $\sum_{i=1}^{W} R_{w-k+1}(b_i) \le \sum_{i=1}^{W} R_{w-k+1}(c_i)$, $M \le c_1 \le \ldots \le c_w \le n+1$. Clearly if $b_w \le n$,

then $\underset{\sim}{c} = \underset{\sim}{a}$ will suffice. Thus, assume $b_w = n+1$.

Let $j = \min_{1 \le k \le w} \{k: b_i \ge a_i\}$. Note $j > 1$, $j \le w$, and $b_{j-1} \le n$. Let

$c_i = a_i I(i < j) + b_i I(i \ge j)$, $d_i = b_i I(i < j) + a_i I(i \ge j)$. Then we have

$$\sum_{i=1}^{W} R_{w-i+1}(c_i) - \sum_{i=1}^{W} R_{w-i+1}(b_i)$$

$$= \sum_{i=1}^{j-1} R_{w-i+1}(a_i) - \sum_{i=1}^{j-1} R_{w-i+1}(b_i)$$

$$= \sum_{i=1}^{W} R_{w-i+1}(a_i) - \sum_{i=1}^{W} R_{w-i+1}(d_i).$$

Since $c_1 = a_1 \ge M$ and $d_w = a_w \le n$, get from (3.23) that

$$\sum_{i=1}^{W} R_{w-i+1}(c_i) - \sum_{i=1}^{W} R_{w-i+1}(b_i) \ge 0.$$

Thus, we have shown by induction (3.20) is true when $C_1^* = C_2^* = \ldots = 0$.

The case where $C_0^* \ge C_1^* \ge \ldots \ge C_w^* \ge 0$ does not follow automatically, due to the peculiar definition of $S_\ell(m)$ in (3.4) necessary to deal with initial distributions. However, as can be easily verified, if indices $\underset{\sim}{a}$ are selected which satisfy (3.23) then for any indices $0 \le b_1 \le \ldots \le b_w \le n$ where $b_1 < M$, by defining

$\underline{c}$ as above, we have that $\sum\limits_{i=1}^{w} P_{w-i+1}(b_i) \leq \sum\limits_{i=1}^{w} P_{w-i+1}(c_i)$, and $c_1 \geq M$. Thus, the lemma is established. $\square$

It should be pointed out that in the above development, Lemma 3.2 was used on the sums $\hat{S}_\ell(n)$, $0 \leq i \leq k$. As stated, this lemma is not applicable. However, since the proof of Lemma 3.2 only requires $n^{-1} \hat{S}_\ell(n) \to E_\pi V_\ell$ a.s. (see Chung [2]), the proof of the theorem is still valid. We now proceed to the proof of theorem 3.1.

PROOF OF THEOREM 3.1. Using the concept of an auxiliary process, with definitions (3.6), (3.7), (3.8) and (3.9), we note that for every $\epsilon > 0$, there exist M_0 and M_1 such that

$$P(T \leq M_0, \sum_{i=1}^{k} Z_i(M_0) \leq M_1) > 1 - \epsilon.$$

Then for $n > M_0$,

$$P(Z_1(n) \leq x_1, \ldots, Z_k(n) \leq x_k, T \leq M_0, \sum_{i=1}^{k} Z_i(M_0) \leq M_1)$$

$$\leq P(Z_1(n) \leq x_1, \ldots, Z_k(n) \leq x_k) \tag{3.24}$$

$$\leq P(Z_1(n) \leq x_1, \ldots, Z_k(n) \leq x_k, T \leq M_0, \sum_{i=1}^{k} Z_i(M_0) \leq M_1) + \epsilon$$

From (2.9), it is easy to see that if $a_i \geq b_i$ for $1 \leq i \leq k$, then

$$P(Z_1(n) \leq x_1, \ldots, Z_k(n) \leq x_k | \underline{Z}(0) = \underline{a})$$

$$\leq P(Z_1(n) \leq x_1, \ldots, Z_k(n) \leq x_k | \underline{Z}(0) = \underline{b}). \tag{3.25}$$

Thus we get

$$P(\hat{Z}_1(n - M_0) \leq x_1, \ldots, \hat{Z}_k(n - M_0) \leq x_k | \hat{Z}_1(0) = M_1, \ldots, \hat{Z}_k(0) = M_1) - \epsilon$$

$$= P(\bar{Z}_1(n - M_0) \leq x_1, \ldots, \bar{Z}_k(n - M_0) \leq x_k | \bar{Z}_1(0) = M_1, \ldots, \bar{Z}_k(M_1)) - \epsilon \qquad (3.26)$$

$$\leq P(Z_1(n) \leq x_1, \ldots, Z_k(n) \leq x_k, T \leq M_0, \sum_{i=1}^{k} Z_i(M_0) \leq M_1).$$

Similarly we can establish that

$$P(Z_1(n) \leq x_1, \ldots, Z_k(n) \leq x_k, T \leq M_0, \sum_{i=1}^{k} Z_i(M_0) \leq M_1)$$

$$\qquad (3.27)$$

$$\leq P(\hat{Z}_1(n - M_0) \leq x_1, \ldots, \hat{Z}_k(n - M_0) \leq x_k | \hat{Z}_1(0) = 0, \ldots, \hat{Z}_k(0) = 0).$$

Thus we get that

$$P(\hat{Z}_1(n - M_0) \leq x_1, \ldots, \hat{Z}_k(n - M_0) \leq x_k | \hat{Z}_1(0) = M_1, \ldots, \hat{Z}_k(0) = M_1) - \epsilon$$

$$\leq P(Z_1(n) \leq x_1, \ldots, Z_k(n) \leq x_k) \qquad (3.28)$$

$$\leq P(\hat{Z}_1(n - M_0) \leq x_1, \ldots, \hat{Z}_k(n - M_0) \leq x_k | \hat{Z}_1(0) = 0, \ldots, \hat{Z}_k(0) = 0) + \epsilon.$$

Since the convergence of $\hat{Z}_1(n)$ is independent of any initial distribution from theorem 3.2, from (3.28) we can see that

$$P(Z_1(n) \leq x_1, \ldots, Z_k(n) \leq x_k) \to P(Z_1 \leq x_1, \ldots, Z_k \leq x_k). \qquad \square$$

4. CONCLUSION

The limit behavior of the compartments was shown to depend on the first moments of the input / output random variables. Since certain compartments diverge, it is reasonable to desire asymptotic behavior of the critical and

supercritical compartments, appropriately normalized. The behavior of these compartments are investigated in (Tollar [12]).

An area of further research would be on the characteristic function of the limiting distribution of the compartments. Any such characterization seems quite difficult, however. For the single cell model, results were obtained by Puri [8], but the techniques used do not seem applicable to the present model.

Another area would be to alter the model to allow a more general flow structure than one - way flow. However, unlike the present model, there appears to be no closed form expression for $Z_i(n)$ in the more general framework. Therefore the use of dual Markov chains will not be applicable to the more general model. It appears that a more general technique using Markov chain theory on arbitrary state spaces may be more fruitful.

REFERENCES

[1]   Balagopal, K. (1979). Some limit theorems for the general semi-Markov storage model, J. Appl. Prob., 16, 607-617.

[2]   Chung, K.L. (1967). Markov Chains with Stationary Transition Probabilities, Springer - Verlag, New York.

[3]   Çinlar, E. (1969). Markov renewal theory, Adv. Appl. Prob., 1, 123-187.

[4]   Hoel, P.G., Port, S.C., and Stone, C.J. (1972). Introduction to Stochastic Processes, Houghton - Mifflin, Boston.

[5]   Lloyd, E.H., and Odoom, S. (1965). A note on the equilibrium distribution of levels in a semi-infinite reservoir subject to Markovian inputs and unit withdrawals, J. Appl. Prob., 2, 215-222.

[6]   Moran, P.A.P. (1954). A probability theory of dams and storage systems, Aust. J. Appl. Sci., 5, 116-124.

[7]   Prabhu, N.U. (1965). Queues and Inventories, J. Wiley and Sons, New York.

[8]   Puri, P.S. (1978). A generalization of a formula of Pollaczek and Spitzer as applied to a storage model, Sankhyā, A, 40, 237-252.

[9]   Puri, P.S. and Senturia, J. (1972). On a mathematical theory of quantal response assays. Proc. Sixth Berkeley Symp. Math. Statist. Prob., 231-247.

[10]   Puri, P.S. and Senturia, J. (1975). An infinite depth dam with Poisson input and Poisson release. Scand. Actuarial J., 193-202.

[11]   Puri, P.S. and Woolford, S.W. (1981). On a generalized storage model with moment assumptions, J. Appl. Prob., 18, 473-481.

[12]   Tollar, E.S. (1985). On the limit behavior of a multi-compartment storage model with underlying Markov chains II: with normalization, FSU Statistics Report M694, February, 1985.

END

FILMED

1-86

DTIC